

# AN EXCEPTIONAL LOCUS IN THE PERFECT COMPACTIFICATION OF $A_g$

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## 1 Introduction

Let  $\mathcal{A}_g$  denote the stack, and  $A_g$  the coarse moduli space, of principally polarized abelian  $g$ -folds and  $\mathcal{A}_g^P$ ,  $A_g^P$  their perfect cone compactifications; these are particular toroidal compactifications. This paper shows that the sublocus  $\mathcal{A}_{1,g-1}^P$  of  $\mathcal{A}_g^P$  that parametrizes abelian varieties with an elliptic factor is exceptional in the sense of birational geometry.

The objects  $\mathcal{A}_g^P$  and  $A_g^P$  exist over  $\mathbf{Spec} \mathbb{Z}$  and there is a contraction  $\pi : \mathcal{A}_g^P \rightarrow A_g^{Sat}$ , the Satake compactification, that factors through  $A_g^P$  [FC]. The main result of [SB] is a description of the cone of curves  $\overline{NE}$  (in Mori's sense) on  $A_g^P$  over any field:  $\overline{NE}(A_g^P)$  is the rational cone spanned by curves  $C_1$  and  $C_2$ . Here  $C_1$  is the closure of the locus of products  $E \times B$ , where  $E$  is a varying elliptic curve and  $B \in A_{g-1}$  is fixed (so that  $C_1$  is a copy of  $A_1^P$ , the compactified  $j$ -line) and  $C_2$  is any curve in the boundary divisor  $D$  that is contracted by the natural morphism  $\pi$  from  $A_g^P$  to the Satake compactification  $A_g^{Sat}$ . In consequence [SB], if  $M$  is the bundle of weight 1 modular forms on  $\mathcal{A}_g^P$  (giving a  $\mathbb{Q}$ -line bundle on  $A_g^P$ ), then  $aM - D$  is ample if and only if  $a > 12$ , and is nef if and only if  $a \geq 12$ .

Recall [Ke] that if  $L$  is a nef divisor class on a projective variety  $X$ , then the *exceptional locus*  $\text{Exc}(L)$  of  $L$  is the union of all the subvarieties  $Z$  of  $X$  with  $L^{\dim Z} \cdot Z = 0$ , and that Keel proved  $L$  is semi-ample on  $X$  provided that it is semi-ample on  $\text{Exc}(L)$  and the ground field  $k$  has  $\text{char } k > 0$ . Notice that Keel's result holds without change when  $X$  is allowed to be any proper stack over a field with finite inertia stack (so that a geometric quotient exists as a proper algebraic space, by [KM]) and projective geometric quotient.

Here is the main result of this paper; I am very grateful to Stefan Schröer, who asked whether  $12M - D$  is semi-ample in positive characteristic.

**Theorem 1.1** (1)  $\text{Exc}(12M - D) = A_{1,g-1}^P$ .

(2) In positive characteristic the bundle  $12M - D$  is semi-ample.

(3) Suppose either that the characteristic is zero and that  $g \leq 11$  or that the characteristic is positive. Then there is a contraction  $A_g^P \rightarrow B_g$  of the ray generated by  $C_1$ , the exceptional locus of the contraction is  $A_{1,g-1}^P$  and the image of the exceptional locus is isomorphic to  $B_{g-1}$ .

## 2 An example: $g = 2$

Here we recall [H, DO] some of the classical geometry of the moduli space  $A_2$  over  $\mathbb{C}$ .

A level 2 structure on a principally polarized abelian surface  $(A, \lambda)$  is a symplectic isomorphism  $\psi : A[2] \rightarrow G := (\mathbb{Z}/2)^2 \times \mu_2^2$ , where  $G$  has its standard symplectic pairing and  $A[2]$  has the Weil pairing defined by the principal polarization  $\lambda$ . There is a standard projective action of  $G$  on  $\mathbb{P}^3$  with homogeneous co-ordinates  $X_{ij}$ , for  $i, j = 0, 1$ ; the  $2\theta$  linear system on  $A$  determined by  $\lambda$  then gives a morphism  $A$  to  $\mathbb{P}^3$  that factors through the Kummer surface  $\text{Km}(A)$ . The image of  $\text{Km}(A)$  lies in a unique  $G$ -invariant quartic; taking the coefficients of this quartic then determines a point on  $\Sigma$ , the Segre cubic threefold [H]; this is the unique cubic threefold with 10 nodes and lies in  $\mathbb{P}^5$  with equations  $e_1 = e_3 = 0$ , where  $e_i$  is the  $i$ th elementary symmetric function in 6 variables. If  $(A, \lambda)$  is irreducible then  $|2\theta|$  is very ample on  $\text{Km}(A)$ , so  $(A, \lambda)$  is determined by the point on  $\Sigma$ , while if  $A = E_1 \times E_2$  then the image of  $\text{Km}(A)$  is  $\mathbb{P}^1 \times \mathbb{P}^1$ , which does not determine  $A$  (“the  $2\theta$  linear system cannot see elliptic factors”).

There are also 15 planes on  $\Sigma$ . Let  $\tilde{\Sigma} \rightarrow \Sigma$  be the blow-up of the nodes,  $E = \sum E_i$  the exceptional divisor of the blow-up and  $H = \sum H_i$  the strict transform of the sum of the planes.

**Theorem 2.1** (1)  $\tilde{\Sigma} = A_{2,2}^P$ , the perfect compactification of the level 2 moduli space  $A_{2,2}$ .  $H$  is the toroidal boundary and  $E$  the locus of products  $E_1 \times E_2$ .

(2)  $A_2^P$  has two distinct birational contractions: one is the standard contraction  $\pi : A_2^P \rightarrow A_2^{\text{Sat}}$  and the other is a contraction  $\rho : A_2^P \rightarrow \Sigma/\mathfrak{S}_6$ , where  $\mathfrak{S}_6$  is the symmetric group on 6 letters and is isomorphic to  $Sp_4(\mathbb{F}_2)$ .

(3)  $\Sigma/\mathfrak{S}_6$  is isomorphic to a weighted projective space  $\mathbb{P}(2, 4, 5, 6)$ ,  $A_{2,2}^{\text{Sat}}$  to the Igusa quartic  $e_1 = e_4 - e_2^2 = 0$  in  $\mathbb{P}^5$  (this is also the projective dual of  $\Sigma$ ) and  $A_2^{\text{Sat}}$  to  $\mathbb{P}(2, 3, 5, 6)$ .

The point of this paper is to extend this picture to all values of  $g$ .

## 3 The structure of the perfect boundary

Let  $C_g$  denote the cone of real positive definite symmetric bilinear forms in  $g$  variables and  $\overline{C}_g$  the cone of real positive semi-definite symmetric bilinear forms in  $g$  variables. Identify  $C_g$  with the interior of  $\overline{C}_g^0$  of  $\overline{C}_g$ . A toroidal compactification of  $\mathcal{A}_g$  or  $A_g$  corresponds to a choice of admissible decomposition of  $\overline{C}_g$ ; we consider here a particular admissible decomposition of  $\overline{C}_g$ , namely, that defined by the perfect quadratic forms. Since our results are particular to this specific toroidal compactification, the arguments depend upon knowing something about perfect forms and the combinatorics of the decomposition of  $\overline{C}_g$  that they define. What we need can be summarized as follows:

**Theorem 3.1** (1) (Voronoi) [AMRT] *The convex hull of the positive semi-definite integral forms in  $\overline{C}_g$  defines a  $GL_g(\mathbb{Z})$ -admissible decomposition of  $\overline{C}_g$ .*

We refer to this admissible decomposition as the *perfect decomposition* and to the cones appearing in it (that is, the facets) as *perfect cones*; the ones of maximal dimension  $g(g+1)/2$  correspond to perfect quadratic forms (this can be taken as the definition of a perfect quadratic form).

(2) (Barnes-Cohn) [BC] *This convex hull coincides with the convex hull of the primitive rank 1 forms. Moreover, a form of rank at least 2 lies in the interior of the hull.*

(3) *If  $p = p(x_1, \dots, x_m)$  and  $q = q(y_1, \dots, y_n)$  are perfect forms of equal minimal norm, then  $p + q = r = r(x_1, \dots, x_m, y_1, \dots, y_n)$  is a form that defines a perfect cone in  $\overline{C}_{m+n}$ .*

(4) *Suppose that  $\tau$  is a perfect cone in  $\overline{C}_r$  that meets  $C_r$ . Then its closure contains perfect cones in some copy of  $\overline{C}_{r-1}$ . In particular, if  $\tau$  is a minimal perfect cone in  $\overline{C}_r$  that meets  $C_r$ , then  $\tau \cap \overline{C}_{r-1}$  is a union of minimal perfect cones in  $\overline{C}_{r-1}$  that meet  $C_{r-1}$ .*

(5) *Suppose that  $\sigma$  is a maximal perfect cone in  $\overline{C}_r$ , so that  $\sigma$  is defined by a perfect form  $q$  in  $r$  variables. Suppose also that  $\tau_1, \tau_2$  are closed cones in  $\overline{C}_{r+1}$  such that both contain  $\sigma$  in their boundary and  $\dim \tau_i = \dim \sigma + 1$ . That is, both cones  $\tau_i$  are minimal with respect to containing  $\sigma$  in their boundary. Then each  $\tau_i$  can be defined by a quadratic form  $\lambda q + l_i^2$ , where  $l_i$  is a primitive linear form and  $\lambda \in \mathbb{Q}$  is the inverse of the minimal norm of  $q$ . In particular,  $\tau_1$  and  $\tau_2$  are conjugate under the parabolic subgroup of  $GL_{r+1}(\mathbb{Z})$  that preserves each of the first  $r$  variables.*

PROOF: For (1) and (2) see the references. (3) is trivial. For (4), we can suppose that  $\sigma$  is minimal with respect to meeting  $C_r$ . Suppose that  $l_1, \dots, l_n$  are primitive elements of  $\Lambda_r^\vee$  such that  $l_1^2, \dots, l_n^2 \in B(\Lambda_r)$  span the 1-dimensional faces of  $\sigma$ . So there are  $\lambda_1, \dots, \lambda_n > 0$  such that  $\sum \lambda_i l_i^2 \in C_r$ . Then, for some  $t$ ,  $\sum_{i=1}^t \lambda_i l_i^2$  is of rank at most  $r-1$  for every  $\lambda_1, \dots, \lambda_t > 0$ , while  $\sum_{i=1}^{t+1} \lambda_i l_i^2$  is of rank  $r$  for some  $\lambda_1, \dots, \lambda_{t+1} > 0$ . Take  $t$  to be maximal subject to this; then the cone  $\tau$  generated by  $l_1^2, \dots, l_t^2$  is a face of  $\sigma$  with the stated properties.

(5) is even easier. □

We denote by  $\mathcal{A}_g^P$  the toroidal compactification of  $\mathcal{A}_g$  that corresponds, as in [FC], to the perfect decomposition of  $\overline{C}_g$ . The boundary  $D = \mathcal{A}_g^P \setminus \mathcal{A}_g$  is a divisor, and is the inverse image  $\pi^{-1}(A_{g-1}^{Sat})$ , where  $A_{g-1}^{Sat}$  is identified with the boundary  $A_g^{Sat} \setminus \mathcal{A}_g$ . We also have the partial compactification  $\mathcal{A}_g^{part}$ , which is open in  $\mathcal{A}_g^{tor}$  and is, by definition, the inverse image  $\pi^{-1}(A_g \amalg A_{g-1})$ . The universal abelian scheme  $\mathcal{U}_{g-r} \rightarrow \mathcal{A}_{g-r}$  has an extension to a semi-abelian scheme  $\mathcal{U}_{g-r}^{part} \rightarrow \mathcal{A}_{g-r}^{part}$  whose degenerate fibres have torus rank 1. Taking  $r$ -fold fibre products gives a semi-abelian scheme  $\delta : \mathcal{U}_{g-r}^{r,part} \rightarrow \mathcal{A}_{g-r}^{part}$ .

In higher codimension the boundary is described as follows. There is a stratified scheme  $F = F_r$ , locally of finite type over the base, the closures of whose

strata are projective toric varieties, with an action of  $GL_r(\mathbb{Z})$  on  $F_r$ , and an  $F_r$ -bundle  $\mathcal{F}_r \rightarrow \mathcal{U}_{g-r}^r$ , with an equivariant action of  $GL_r(\mathbb{Z})$ , such that  $\pi^{-1}(A_{g-r}) = \mathcal{F}_r/GL_r(\mathbb{Z})$ , the stack quotient. The closures  $F_{r,\tau}$  of the various strata of  $F_r$  correspond to the perfect cones  $\tau$  in the perfect decomposition of  $\overline{C}_r$  that meet the interior  $C_r$ . In particular, the irreducible components of  $F_r$  correspond to the minimal such cones.

Here is a more precise description of  $F_r$ : put  $\Lambda_r = \mathbb{Z}^{\oplus r}$ , let  $B_r$  be the lattice of symmetric bilinear forms on  $\Lambda_r$  and denote by  $T_r$  the torus with character group  $\mathbb{X}^*(T_r) = B_r$ . Then there is a locally finite  $GL_r(\mathbb{Z})$ -equivariant torus embedding  $T_r \hookrightarrow Y_r$  such that  $F_r$  is a  $T_r \rtimes GL_r(\mathbb{Z})$ -equivariant closed subscheme of the boundary  $Y_r \setminus T_r$ . The closure  $F_{r,\tau}$  of a stratum in  $F_r$  is then a torus embedding under a quotient  $T$  of  $T_r$  and gives rise to the closure  $\mathcal{F}_{r,\tau}$  of a stratum in  $\mathcal{F}_r$ ; this closure is a proper  $F_{r,\tau}$ -bundle  $\mathcal{F}_{r,\tau} \rightarrow \mathcal{U}_{g-r}^r$  that is a relative  $T$ -equivariant compactification of a  $T$ -bundle  $\mathcal{T} \rightarrow \mathcal{U}_{g-r}^r$ .

In turn, the image of  $\mathcal{F}_{r,\tau}$  in  $\mathcal{A}_g^P$  is an irreducible closed substack  $\mathcal{X}_{r,\tau}$  of  $\pi^{-1}(A_{g-r})$ . If  $n \geq 3$  and is invertible in the base, then in the stack  $\mathcal{A}_{g,n}^P$  the image  $\mathcal{X}_{r,\tau}$  can be identified with  $\mathcal{F}_{r,\tau}$ . In this case (that is, at level  $n$ )  $T_r$  acts on  $\mathcal{X}_{r,\tau}$ , via the quotient  $T_r \rightarrow T$ , and this action extends to an action on the closure  $\overline{\mathcal{X}}_{r,\tau}$  of  $\mathcal{X}_{r,\tau}$  in  $\mathcal{A}_{g,n}^P$ .

Each such  $\tau$  lies in the closure of finitely many maximal perfect cones  $\sigma$  in  $\overline{C}_r$ . Such a cone  $\sigma$  corresponds to the choice, up to scalars, of a perfect form  $q$  in  $r$  variables. The closure  $\overline{\mathcal{U}}_{g-r,\sigma}^r$  of  $\mathcal{U}_{g-r}^r$  is just  $\overline{\mathcal{X}}_{r,\sigma}$ . We let  $\overline{\mathcal{U}}_{g-r,\sigma}^{r,norm}$  denote the normalization of  $\overline{\mathcal{U}}_{g-r,\sigma}^r$ .

Translating Theorem 3.1 into algebraic geometry yields the following statement.

**Corollary 3.2** (1) (Corollary of 3.1 1) *There exist toroidal compactifications  $\mathcal{A}_g^P$  and  $\mathcal{A}_g^P$  corresponding to this decomposition.*

(2) (Corollary of 3.1 2) *As a Deligne-Mumford stack,  $\mathcal{A}_g^P$  has terminal singularities and the boundary  $D$  is absolutely irreducible.*

(3) (Corollary of 3.1 3) *The product morphism  $\mathcal{A}_g \times \mathcal{A}_h \rightarrow \mathcal{A}_{g+h}$  extends to a morphism  $\mathcal{A}_g^P \times \mathcal{A}_h^P \rightarrow \mathcal{A}_{g+h}^P$ .*

(4) (Corollary of 3.1 4)  *$\pi^{-1}(A_{g-r}^{Sat})$  lies in the closure of  $\pi^{-1}(A_{g-r+1})$  and  $\pi^{-1}(A_{g-r+1}^{Sat})$  is the closure of  $\pi^{-1}(A_{g-r+1})$ .*

(5) (Corollary of 3.1 5) *A maximal perfect cone  $\sigma$  in  $\overline{C}_r$  corresponds to an irreducible closed substack  $\overline{\mathcal{U}}_{g-r,\sigma}^r$  of  $\pi^{-1}(A_{g-r}^{Sat})$  that contains an open substack isomorphic to  $\mathcal{U}_{g-r}^{r,part}$ . The complement  $\overline{\mathcal{U}}_{g-r,\sigma}^r \setminus \mathcal{U}_{g-r}^{r,part}$  has codimension at least 2 in  $\overline{\mathcal{U}}_{g-r,\sigma}^r$ .*

PROOF: (1), (2) and (3) are immediate.

For (4), we use the fact that the irreducible components  $\mathcal{Z}$  of  $\pi^{-1}(A_{g-r})$  correspond to the (equivalence classes of the) minimal perfect cones  $\tau$  in  $\overline{C}_r$  that meet  $C_r$ . Since, by 3.1 4,  $\tau \cap \partial\overline{C}_r$  is a union of minimal perfect cones in

$\overline{C}_{r-1}$  that meet  $C_{r-1}$ , the closure  $\overline{\mathcal{Z}}$  of  $\mathcal{Z}$  lies in the closure of an irreducible component of  $\pi^{-1}(A_{g-r+1})$ . So  $\pi^{-1}(A_{g-r}) \subseteq \overline{\pi^{-1}(A_{g-r+1})}$ . Then, by induction,  $\pi^{-1}(A_{g-r-m}) \subseteq \overline{\pi^{-1}(A_{g-r+1})}$ , so that  $\pi^{-1}(A_{g-r}^{Sat}) \subseteq \overline{\pi^{-1}(A_{g-r+1})}$ . Therefore  $\pi^{-1}(A_{g-r+1}) = \overline{\pi^{-1}(A_{g-r+1})}$ .

For (5), note that the irreducible components of  $\overline{\mathcal{U}}_{g-r,\sigma}^r \setminus \mathcal{U}_{g-r}^{r,part}$  correspond to the minimal perfect cones  $\tau$  in  $\overline{C}_{r+1}$  that contain  $\sigma$ . These are equivalent, and we are done.  $\square$

**Definition 3.3** *An open substack  $\mathcal{U}$  of an algebraic stack  $\mathcal{X}$  is nearly equal (abbreviated to n.e.) to  $\mathcal{X}$  if its complement has codimension at least 2 everywhere.*

In particular,  $\mathcal{A}_g^{part}$  is nearly equal to  $\mathcal{A}_g^P$ .

## 4 The exceptional locus of $12M - D$

Set  $L_g = 12M - D_g$ .

**Proposition 4.1** *For sufficiently divisible  $N$ , the linear system  $|NL_g|$  has no base points in  $\mathcal{A}_g^{part}$  and contracts all curves (that is, complete 1-dimensional substacks) of the form  $\mathcal{A}_1^P \times \{B\}$  with  $B \in \mathcal{A}_{g-1}$ . This morphism separates points except along  $\mathcal{A}_{1,g-1}^P \cap \mathcal{A}_g^{part}$ .*

PROOF: First, work over  $\mathbb{Z}[1/2]$  and impose a level 2 structure. and considering the morphism defined by the  $2\Theta$  linear system: there is a universal family  $f : V \rightarrow \mathcal{A}_{g,2}^{part}$  of projective schemes with level 2 structure, and the  $2\Theta$  linear system defines, by taking the cycle-theoretic image of each fibre of  $f$ , a morphism  $\Phi$  from  $\mathcal{A}_{g,2}^{part}$  to the Chow scheme of  $\mathbb{P}^{2g-1}$ . After dividing by the finite group  $Sp_{2g}(\mathbb{Z}/2)$  we then get a morphism  $\phi$  from  $\mathcal{A}_g^{part}$  to some scheme, and  $\phi$  contracts every curve of the form  $\mathcal{A}_1^P \times \{B\}$ , since taking the Kummer variety of an elliptic curve collapses the  $j$ -line to a point. It follows that  $\phi$  is defined by some linear system  $|NL|$ , since when  $g = 1$  the bundle  $M$  has degree  $1/12$  and  $D$  has degree 1.

Over  $\mathbb{Z}[1/3]$  we work at level 3, and then look at the Grassmannian that parametrizes the spaces of quadrics that pass through the  $3\Theta$  image. Since a degree 3 elliptic curve cannot be recovered from the quadrics that contain it, this performs a similar function of “losing elliptic factors”.  $\square$

So pick  $r \geq 2$  and consider  $L_g$  on the inverse image  $\pi^{-1}(A_{g-r})$ , where  $A_{g-r}$  is regarded as a stratum in  $A_g^{Sat}$ . The closure  $\mathcal{Z}_r$  of  $\pi^{-1}(A_{g-r})$  in  $\mathcal{A}_g^P$  is a finite union of irreducible components  $\overline{\mathcal{X}}_{r,\tau}$  as above. Note that  $\mathcal{Z}_r = \pi^{-1}(A_{g-r}^{Sat})$ , as already pointed out.

Fix a maximal perfect cone  $\sigma$  in  $\overline{C}_r$  and a face  $\tau$  of  $\sigma$  that meets  $C_r$ . There is, up to scalars, a unique perfect quadratic form  $q$  which defines  $\sigma$  and can be written, in many ways, as a linear combination  $q = \sum_1^r \lambda_i x_i^2$  where  $x_i$  is a primitive integral linear form and  $\lambda_i \in \mathbb{Q}_+$ . Each rank 1 form  $x_i^2$  corresponds, in  $\text{Hom} \otimes \mathbb{Q}$ , to a projection  $\pi_i : \mathcal{U}_{g-r}^r \rightarrow \mathcal{U}_{g-r}$  over  $\mathcal{A}_{g-r}$ . So (since homomorphisms

of abelian schemes extend uniquely to homomorphisms of semi-abelian schemes) there is a diagram

$$\begin{array}{ccccccc}
 \overline{\mathcal{X}}_{r,\tau} & \xleftarrow{\text{closed}} & \overline{\mathcal{U}}_{g-r,\sigma}^r & \xleftarrow{n.e.} & \mathcal{U}_{g-r}^{r,\text{part}} & \xrightarrow{\pi_i} & \mathcal{U}_{g-r}^{\text{part}} \xrightarrow{n.e.} D_{g-r+1} \xrightarrow{\text{closed}} \mathcal{A}_{g-r+1}^P \\
 \downarrow \text{closed} & & \searrow \alpha_{g,r} & & \downarrow \delta & \swarrow \gamma & \\
 \mathcal{A}_g^P & & & & \mathcal{A}_{g-r}^{\text{part}} & & 
 \end{array}$$

(Here the labels *closed* and *n.e.* refer to closed embeddings and near equalities.)

**Proposition 4.2** *Let  $s$  denote the 0-section of  $\delta$ . Then the restrictions  $\alpha_{g,r}^* D_g|_s$  and  $\delta^* D_{g-r}|_s$  are linearly equivalent.*

PROOF: Recall that the multiplication  $\mathcal{A}_{g-r} \times \mathcal{A}_r \rightarrow \mathcal{A}_g$  extends to  $\mathcal{A}_{g-r}^P \times \mathcal{A}_r^P \rightarrow \mathcal{A}_g^P$ . The perfect form  $q$  corresponds to a maximally degenerate boundary point  $x \in \mathcal{A}_r^P$ , and then the image of  $\{x\} \times \mathcal{A}_{g-r}^P$  in  $\mathcal{A}_g^P$  is exactly the closure in  $\mathcal{A}_g^P$  of the zero section of  $\mathcal{U}_{g-r}^r \rightarrow \mathcal{A}_{g-r}$ .

Identify  $s$  with  $\{x\} \times \mathcal{A}_{g-r}^{\text{part}}$  as above. Then there is a chain of rational curves in  $\mathcal{A}_r^P$  leading from  $x$  to an interior point  $y = E^r$ , for any elliptic curve  $E$  (let  $E$  degenerate, and then take a rational chain leading from this maximally degenerate boundary point to  $y$ ).

So  $D_g|_s$  is rationally equivalent to  $D_g|_{\{y\} \times \mathcal{A}_{g-r}^{\text{part}}}$ , and now the result is obvious.  $\square$

Set  $\Lambda = -\alpha_{g,r}^* D_g + \delta^* D_{g-r}$ , a divisor class on  $\mathcal{U}_{g-r}^{r,\text{part}}$ . Since  $\mathcal{U}_{g-r}^{r,\text{part}} \rightarrow \mathcal{A}_{g-r}^{\text{part}}$  is semi-abelian and  $\Lambda$  is trivial on the zero section,  $\Lambda$  is determined by the polarization that it defines on the generic fibre  $\mathcal{U}_{g-r,\eta}^r$ . (At this point we also use the fact that line bundles on the semi-abelian scheme  $\mathcal{U}_{g-r}^{r,\text{part}} \rightarrow \mathcal{A}_{g-r}^{\text{part}}$  that are trivial on the zero section are determined by the polarization that they define on the generic fibre. This is because the only global point of the self-dual abelian scheme  $\mathcal{U}_{g-r}^r \rightarrow \mathcal{A}_{g-r}$  is zero.)

**Corollary 4.3**

$$\Lambda \sim \sum_1^r \lambda_i \pi_i^* \left( -D_{g-r+1}|_{\mathcal{U}_{g-r}^{\text{part}}} + \gamma^* D_{g-r} \right).$$

PROOF: From its definition, and knowledge of the polarization defined by  $-D_g$  on  $\mathcal{U}_{g-r,\eta}^r$ , the polarization defined by  $\Lambda$  on  $\mathcal{U}_{g-r,\eta}^r$  is the quadratic form  $q$ . The polarization defined by  $-D_{g-r+1}$  on the generic fibre of  $\mathcal{U}_{g-r}$  is that given by the primitive rank one form  $x_i^2$ , and we are done.  $\square$

For  $i = 1, \dots, r$ , choose a large positive integer  $n_i$  and set  $\tilde{\pi}_i = [n_i] \circ \pi_i : \mathcal{U}_{g-r}^{r,\text{part}} \rightarrow \mathcal{U}_{g-r}^{\text{part}}$  and  $\rho_i = \alpha_{g-r+1,1} \circ \tilde{\pi}_i : \mathcal{U}_{g-r}^{r,\text{part}} \rightarrow \mathcal{A}_{g-r+1}^P$ . Then

$$\alpha_{g,r}^* \Lambda \sim \sum_1^r \frac{\lambda_i}{n_i^2} \tilde{\pi}_i^* \left( -D_{g-r+1}|_{\mathcal{U}_{g-r}^{\text{part}}} + \gamma^* D_{g-r} \right),$$

which can be re-written as

$$L_g|_{\mathcal{U}_{g-r}^{r,part}} \sim \left(1 - \sum_i \frac{\lambda_i}{n_i^2}\right) \delta^* L_{g-r} + \sum_i \frac{\lambda_i}{n_i^2} \rho_i^* L_{g-r+1}.$$

We abbreviate this to

$$(*) \quad L_g|_{\mathcal{U}_{g-r}^{r,part}} \sim a \delta^* L_{g-r} + \sum b_i \rho_i^* L_{g-r+1}.$$

Note that  $a, b_i > 0$ .

**Theorem 4.4** (= Theorem 1.1) (1)  $L_g$  is nef.

(2)  $\text{Exc}(L_g) = \mathcal{A}_{1,g-1}^P$  and, over a field of positive characteristic,  $L_g$  is semi-ample.

(3) Suppose either that the characteristic is zero and that  $g \leq 11$  or that the characteristic is positive. Then there is a contraction  $A_g^P \rightarrow B_g$  of the ray  $R_1$ , the exceptional locus of the contraction is  $A_{1,g-1}^P$  and the image of the exceptional locus is isomorphic to  $B_{g-1}$ .

PROOF: (1) was proved in [SB]. We give a proof here that can be carried over to prove (2) also.

Assume that  $L_g$  is not nef, and that  $\mathcal{C}$  is a complete curve in  $\mathcal{A}_g^P$  with  $L_g \cdot \mathcal{C} < 0$ . By Proposition 4.1, and since  $-D_g$  is  $\pi$ -ample, there is some minimal  $r \geq 2$  such that  $\mathcal{C}$  maps to a curve in  $A_{g-r}^{Sat}$  that does not lie in  $A_{g-r-1}^{Sat}$ .

Consider  $L_g$  on the inverse image  $\pi^{-1}(A_{g-r})$ , where  $A_{g-r}$  is regarded as a locally closed subvariety of  $A_g^{Sat}$ . The closure  $\mathcal{Z}^r$  of  $\pi^{-1}(A_{g-r})$  in  $\mathcal{A}_g^P$  is a finite union of irreducible components  $\overline{\mathcal{X}}_{r,\tau}$ , each of which is the image of an equivariant closure  $\overline{\mathcal{X}}_{r,\tau,n}$  in  $\mathcal{A}_{g,n}^P$  of a  $T$ -bundle, where  $T$  is a quotient of the torus  $T_r$  with cocharacter group  $B_r$ .

We can use the  $T$ -action at level  $n$  to construct a specialization (that is, a rational equivalence)  $\mathcal{C} \sim \mathcal{C}_0 + \mathcal{F}$ , where  $\mathcal{C}_0$  is contained in a minimal stratum of  $\mathcal{Z}_r$  and  $\mathcal{F}$  is the image of a closed subvariety  $\mathcal{F}_n$  in  $\mathcal{A}_{g,n}^P$  such that  $\mathcal{F}_n$  is preserved by  $T$  but no component of  $\mathcal{F}_n$  consists of fixed points. So then  $\mathcal{F}$  is  $\pi$ -vertical, so that  $(-D) \cdot \mathcal{F} > 0$  and  $M \cdot \mathcal{F} = 0$ , and then  $\mathcal{C}_0$  is  $L$ -negative. So we can assume that  $\mathcal{C}$  lies in a minimal stratum of  $\mathcal{Z}_r$ . Recall that each such stratum is one of the closed substacks  $\overline{\mathcal{U}}_{g-r,\sigma}^r$  considered previously.

Since  $\mathcal{U}_{g-r}^{r,part}$  is nearly equal to  $\overline{\mathcal{U}}_{g-r,\sigma}^{r,norm}$ , the linear equivalence  $(*)$  on  $\mathcal{U}_{g-r}^{r,part}$  extends to  $\overline{\mathcal{U}}_{g-r,\sigma}^{r,norm}$ . Since  $L_h$  has (stably) no base points on  $\mathcal{A}_h^{part}$ , by Proposition 4.1, it follows that  $\mathcal{C}$  is disjoint from the open substack  $\mathcal{U}_{g-r}^{r,part}$ . But this contradicts the assumption that  $\pi(\mathcal{C})$  does not lie in  $A_{g-r-1}^{Sat}$ , and (1) is proved.

For (2), assume first that we are in positive characteristic.

Suppose that  $\mathcal{Z}$  is an irreducible closed substack of  $\mathcal{A}_g^P$  with  $\mathcal{Z} \cdot L_g^{\dim \mathcal{Z}} = 0$ , so that  $\mathcal{Z}$  lies in  $\text{Exc}(L_g)$ . If  $\mathcal{Z} \cap \mathcal{A}_g^{part}$  is not empty, then  $\mathcal{Z}$  lies in  $\mathcal{A}_{1,g-1}^P$ , as desired. So we can suppose that  $\mathcal{Z}$  lies over  $A_{g-r}^{Sat}$ , but not over  $A_{g-r-1}^{Sat}$ , where  $r \geq 2$ . Then  $\mathcal{Z}$  lies in some  $\overline{\mathcal{X}}_r$ .

Note first that  $r < g$ , since  $L_g$  is ample on the fibre  $\pi^{-1}(A_0^{Sat})$  of  $\pi$ . (This is the statement that  $-D_g$  is  $\pi$ -ample, which holds because, as a toridal compactification,  $\mathcal{A}_g^P$  is defined by taking a convex hull.) Also, we assume, as an induction hypothesis, that  $L_h$  is semi-ample on  $\mathcal{A}_h^P$  for all  $h < g$ .

Once again we can use a torus action to construct a rational equivalence  $\mathcal{Z} \sim \mathcal{Z}_0 = \mathcal{Y} + \mathcal{W}$  where  $\mathcal{Y}$  lies in  $\overline{\mathcal{U}}_{g-r,\sigma}^r$  and the fibres of  $\mathcal{W} \rightarrow A_g^{Sat}$  are of strictly positive dimension.

Suppose first that  $\mathcal{Z} = \mathcal{Y}$ , i.e., that  $\mathcal{Z}$  lies in some  $\overline{\mathcal{U}}_{g-r,\sigma}^r$ . Since  $L_g$  is nef, we have  $\mathcal{Z}.L_g^{\dim \mathcal{Z}} = 0$ . Put  $\mathcal{Z}^0 = \mathcal{Z} \cap \mathcal{U}_{g-r}^{r,part}$ ; this is open and dense in  $\mathcal{Z}$ .

From the linear equivalence (\*) it follows that the stable base locus of the restriction  $L_g|_{\overline{\mathcal{U}}_{g-r,\sigma}^r}$  lies in the boundary  $\overline{\mathcal{U}}_{g-r,\sigma}^r \setminus \mathcal{U}_{g-r}^{r,part}$ . Therefore, by Kodaira's lemma ([Ko] VI.2.15, VI.2.16), the restriction  $L'$  of  $L_g$  to  $\mathcal{U}_{g-r}^{r,part}$  is semi-ample but not big (i.e., a large multiple of  $L'$  defines a morphism on  $\mathcal{Z}^0$  but that morphism is not birational to its image), so that  $\mathcal{Z}^0$  is covered by open curves  $\mathcal{C}^0$  on which the morphism defined by the linear system  $|nL'|$ , for some suitable  $n$ , is constant.

Then, for each  $i$ , the linear system  $|nL_{g-r+1}|$  defines a constant morphism on each curve  $\rho_i(\mathcal{C}^0)$ . By induction,  $|nL_{g-r+1}|$  has no base points on  $\mathcal{A}_{g-r+1}^P$ , so the closure of  $\rho_i(\mathcal{C}^0)$  lies in  $\text{Exc}(L_{g-r+1})$ .

By the induction hypothesis,  $\text{Exc}(L_{g-r+1}) = \mathcal{A}_{1,g-r}^P$ . Moreover, if  $\mathcal{U}_{g-r}^{part}$  is identified with an open substack of the boundary  $D_{g-r+1}$  of  $\mathcal{A}_{g-r+1}^P$ , then taking the  $j$ -invariant of the elliptic factor to be  $\infty$  shows that  $\text{Exc}(L_{g-r+1}) \cap \mathcal{U}_{g-r}^{part}$  contains the closure of the zero-section of the semi-abelian scheme  $\mathcal{U}_{g-r}^{part} \rightarrow \mathcal{A}_{g-r}^{part}$ .

Consider the intersection  $\mathcal{I} = \mathcal{A}_{1,g-r}^P \cap \mathcal{U}_{g-r}^{part}$ , taken inside  $D_{g-r+1}$ .

**Lemma 4.5**  *$\mathcal{I}$  has just two irreducible components. One is the locus of points of the form*

$$(\infty, B, 0_B),$$

*where  $B \in \mathcal{A}_{g-r}$  and  $\infty$  is the point at infinity on  $\mathcal{A}_1^P$ ; this is a copy of  $\mathcal{A}_{g-r}$ . The other is the locus of points of the form*

$$(E \times V, (0_E, v)),$$

*where  $E \in \mathcal{A}_1$ ,  $V \in \mathcal{A}_{g-r-1}$  and  $v \in V$  is arbitrary; this is a copy of  $\mathcal{A}_1^P \times \mathcal{U}_{g-r-1}$ .*

PROOF: The only thing to notice is that on  $\mathcal{A}_{g-r+1}^{part}$ , the exceptional locus  $\text{Exc}(L_{g-r+1})$  includes the image of  $\mathcal{A}_1 \times \mathcal{A}_{g-r}^{part}$ , the locus where the cycle-theoretic image of the Kummer variety under the  $2\Theta$  linear system does not determine the abelian variety. The pair  $(V, v)$  corresponds to a compactification  $\tilde{V}$  of some  $\mathbb{G}_m$ -bundle over  $V$ , and then taking the Kummer variety of  $E \times \tilde{V}$  has the effect of “losing the isomorphism class of  $E$ ”.  $\square$

That is, for every  $\tilde{\pi}_i$ , the image  $\tilde{\pi}_i(\mathcal{Z})$  is contained in the union of these two loci. Now consider the summand  $a\delta^*L_{g-r}$  that appears as a contribution



to  $L_g|_{\mathcal{U}_{g-r}^{r,part}}$  in  $(*)$ ; since, by induction,  $L_{g-r}$  is semi-ample and  $\text{Exc}(L_{g-r}) = \mathcal{A}_{1,g-r-1}^P$ , this consideration shows that

$$\mathcal{Z}^0 \cap \mathcal{U}_{g-r}^r \subset \mathcal{A}_1 \times \mathcal{U}_{g-r-1}^r.$$

Now  $\mathcal{Z}$  lies in the closure of  $\mathcal{Z}^0 \cap \mathcal{U}_{g-r}^r$  in  $\overline{\mathcal{U}}_{g-r,\sigma}^r$ , so that  $\mathcal{Z}$  is in (the image of)  $\mathcal{A}_1^P \times \overline{\mathcal{U}}_{g-r-1,\sigma}^r$  in  $\mathcal{A}_{1,g-1}^P$ . In particular,  $\mathcal{Z}$  lies in  $\mathcal{A}_{1,g-1}^P$ .

Now drop the assumption that  $\mathcal{Z}$  lies in some  $\overline{\mathcal{U}}_{g-r,\sigma}^r$ . Then  $\mathcal{Z}$  specializes as above to  $\mathcal{Z}_0 = \mathcal{W} + \mathcal{Y}$  where  $\mathcal{Y}$  lies in some  $\overline{\mathcal{U}}_{g-r,\sigma}^r$ . Since  $L_g$  is nef, both  $\mathcal{W}$  and  $\mathcal{Y}$  lie in  $\text{Exc}(L_g)$ , so that, by what we have already proved,  $\mathcal{Y}$  lies in  $\mathcal{A}_{1,g-1}^P$ .

Recall that  $\mathcal{Z}$  lies in  $\overline{\mathcal{X}}_{r,\tau}$ , the image of the closure of a  $T$ -bundle  $\mathcal{T} \rightarrow \mathcal{U}_{g-r}^r$ . The specialization  $\mathcal{Z} \sim \mathcal{Z}_0$  and the fact that  $\mathcal{Y}$  lies in  $\mathcal{A}_{1,g-1}^P$  show that  $\mathcal{Z}$  lies in the image of the closure of the restriction of  $\mathcal{T}$  to the closed substack  $\mathcal{A}_1 \times \mathcal{U}_{g-r-1}^r$  of  $\mathcal{U}_{g-r}^r$ . But this restriction is of the form  $\mathcal{A}_1 \times \mathcal{T}_1$ , where  $\mathcal{T}_1$  is a  $T$ -bundle over  $\mathcal{U}_{g-r-1}^r$ . So  $\mathcal{Z}$  lies in  $\mathcal{A}_1^P \times \overline{\mathcal{X}}'_{r,\tau}$ , where  $\overline{\mathcal{X}}'_{r,\tau}$  is the image of the closure of  $\mathcal{T}_1$ . However,  $\overline{\mathcal{X}}'_{r,\tau}$  lies in  $\mathcal{A}_{1,g-1}^P$ , so that  $\mathcal{Z}$  lies in  $\mathcal{A}_{1,g-1}^P$ , as required.

That is, we have shown that in characteristic  $p > 0$ ,  $\text{Exc}(L_g) \subset \mathcal{A}_{1,g-1}^P$ ; the other inclusion is an immediate consequence of the fact that  $L_1$  is trivial. From Keel's theorem we deduce that  $L_g$  is semi-ample. It follows at once that  $\text{Exc}(L_g) = \mathcal{A}_{1,g-1}^P$  in characteristic zero, and now (2) is proved.

(3) follows from the standard theorems in birational geometry in characteristic zero, and from Keel's result in positive characteristic.  $\square$

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